Improvement of the Discrete Least Squares Polynomial Approximation for Stochastic Elliptic PDEs via a Reduced Basis Method

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Introduction

In the fields of science and engineering mathematical models are utilized to understand and predict the behavior of complex systems. Common input data/parameters for these type of models include forcing terms, boundary conditions, model coefficients and the computational domain itself. Often times for any number of reasons there is a degree of uncertainty involved with these different types of inputs. In order to obtain an accurate model we must incorporate this uncertainty into the governing equations, and quantify its effects on the output of the simulation. In this work we explore the recently analyzed discrete least squares method and the application of a reduced basis method to cut down on the computational cost.

Problem Setting

We focus on solving the stochastic elliptic problem and the corresponding discrete problem: find $u : \Omega \times D \to \mathbb{R}$ such that it holds almost surely:

$$\begin{aligned} -\nabla(a(x,\boldsymbol{y})\nabla u(x,\boldsymbol{y})) &= f \quad \forall (x,\boldsymbol{y}) \in D \times \Gamma \\ u(x,\boldsymbol{y}) &= 0 \quad \forall (x,\boldsymbol{y}) \in D \times \Gamma. \end{aligned} \tag{1}$$

with the following assumption on the diffusion coefficent

- The coefficient $a(x,\boldsymbol{y})$ can be written in the form

$$a(x,\boldsymbol{y}) = a_0(x) + \sum_{k=1}^{N} a_k(x) y_k(\omega). \tag{2}$$

- Defining L^2_φ to be the space of square integrable functions on Γ with respect to the measure φ, $V = H^1_0(D)$ and V' be the corresponding dual space we can express the corresponding weak formulation.
- Find $u(x,\boldsymbol{y}) \in V \otimes L^2_\varphi(\Gamma)$ such that:

$$\int_\Gamma \int_D a(x,\boldsymbol{y}) \nabla u(x,\boldsymbol{y}) \cdot \nabla v(x,\boldsymbol{y}) \varphi(\boldsymbol{y}) dx d\boldsymbol{y} = \int_\Gamma \int_D f(x) v(x,\boldsymbol{y}) \varphi(\boldsymbol{y}) dx d\boldsymbol{y} \quad \forall v \in V \otimes L^2_\varphi(\Gamma). \tag{3}$$

- We are interested in some statistical Quantity of Interest (QOI) related to the solution of (3), some examples would be the spatial mean or variance of the solution.

Least Squares in Hilbert Spaces

- We wish to find the best approximation of $u \in V \otimes L^2_\varphi(\Gamma)$ in the discrete least squares sense.
- We let $\{\ell_j\}_{j=1}^{\#\Lambda}$ denote a polynomial space, $\{\psi_j\}_{j=1}^{N_h}$ denote a finite element basis and we define

$$D_{ij} = \ell_j(\boldsymbol{y}_i). \tag{4}$$

- We then seek the discrete least squares approximation

$$u_\Lambda = P^n_m u = \sum_{\mathbf{s} \in \Lambda} \sum_{j=1}^{N_h} c_{\mathbf{s}j} \ell_{\mathbf{s}}(\mathbf{y}) \psi_j(\mathbf{x}). \tag{5}$$

where $C = \{c_{\mathbf{s}j}\}$ satisfies

$$(D^T D) C = D^T U \tag{6}$$

$u(\mathbf{y}_i) = \sum_{j=1}^{N_h} u_j(\mathbf{y}^{(i)}) \psi_j$, and $U_{ij} = u_j(\mathbf{y}^{(i)})$.

- This will then decouple into N_h least squares problems

$$D^T D C_{:,j} = D^T U_{:,j}. \tag{7}$$

Reduced Basis Method

- Letting $X = V \otimes L^2_\varphi(\Gamma)$ we can rewrite (1) in the weak form

$$A(u(y), v; y) = f(v) \quad \forall v \in X \tag{8}$$

and the associated Galerkin projection problem as:

$$A(u_{\mathcal{N}}(y), v; y) = f(v) \quad \forall v \in X_{\mathcal{N}} \tag{9}$$

where $X_{\mathcal{N}}$ is a linear subspace of dimension $\mathcal{N} >> 1$ (i.e. FEM space).

- A reduced basis approach seeks to cut down on the cost of each individual realization of (9) by constructing a linear subspace of $X_{\mathcal{N}}$

$$X_{\mathcal{N},N} = \text{Span}(u_{\mathcal{N}}(y^N_n)) \quad n = 1, ..., N \tag{10}$$

where $N << \mathcal{N}$, that serves as a good approximation to our finite element space.

- Using a greedy algorithm with an online/offline decomposition it is possible to obtain massive cost savings while still maintaining the accuracy of our calculations.
- Two key components of this algorithm are the training set $\Xi_{\text{train}} \subset \Gamma$ we select snapshots from, and the a posterior error estimator $\triangle^u_N(y)$ which we use to validate that our reduced basis is indeed a good approximant.
- A rough outline of the greedy algorithm reads :

1: Begin offline portion
2: randomly select $y^1 \in \Xi_{\text{train}}$
3: compute $u_{\mathcal{N}}(y^1)$ and initialize the reduced basis $X_{\mathcal{N},1} = \text{Span}(u_{\mathcal{N}}(y^1))$
4: for $n = 2$ till N_{max}
5: $y^n = \text{argmax}\{\triangle^u_{n-1}(y), y \in \Xi_{\text{train}}\}$
6: compute $u_{\mathcal{N}}(y^n)$ and let $X_{\mathcal{N},m} = \text{Span}(u_{\mathcal{N}}(y^m)), m = 1, ..., n$
7: end for

- Alternatively to terminating the algorithm when we reach some level N_{max} we can preset some error tolerance ϵ_{rom} and end the algorithm when the approximation error is judged to be sufficiently small

Incorporation in the least squares algorithm

- We can split the error into the sums of a spatial discretization error, least squares approximation error, and reduced basis error, i.e.

$$\mathbb{E}(||u - P^n_m u^r_h||^2) \leq \underbrace{||u - u_h||^2_X}_{I} + \underbrace{||u_h - u^r_h||^2_X}_{II} + \underbrace{\mathbb{E}(||u^r_h - P^n_m u^r_h||^2_X)}_{III}. \tag{11}$$

- Our strategy will be to drive the reduced basis error to a level ϵ using the a posteriori error estimate $\triangle^u_N$. We will then determine the number of sample points n and the finite element discretization level h so that they are of the same level as the reduced basis error.

A rough outline of our algorithm reads :
1: Set the desired error tolerance for the reduced basis method ϵ_{rom}
2: Using ϵ_{rom} determine the appropriate FEM discretization h
3: Execute the offline portion of the reduced basis algorithm
4: Using h and error estimates for II in (11) determine the appropriate number of sample points n for the least squares problem and the associated polynomial space
5: Solve the least squares system (7) and calculate the associated QOI

- Analyzing the computational cost we have that the full least squares method will scale as

$$LS_{\text{full}} = n \times O\left(\frac{1}{h^\alpha}\right) + O(m^3) + O(m^2) \times \frac{1}{h}. \tag{12}$$

- Here n is the total number of sample points, h is the mesh spacing parameter, $O\left(\frac{1}{h^\alpha}\right)$ is the cost for solving the finite element system where α depends on both the solver and spatial dimension, $O(m^3)$ is the cost associated with the LU or QR decomposition, and $O(m^2) \times \frac{1}{h}$ is the cost for solving the system (7).
- analyzing the algorithm with the reduced basis incorporated into it. The offline portion of the algorithm will scale as

$$RB_{\text{offline}} = O(n_{\text{train}}) \times (\sum_{\ell=1}^{N_{\text{red}}-1} w_{\text{online}}(\ell)) + N_{\text{red}} \times O\left(\frac{1}{h^\alpha}\right). \tag{13}$$

- Where $O(n_{\text{train}})$ is the cost of a max search in our training set, and w_{online} is the cost for calculating $\triangle^u_n(y)$ and $u_{\mathcal{N},n}(y)$ for a value $y \in \Xi_{train}$.
- The total cost for our algorithm will thus scale as

$$RB_{\text{total}} = RB_{\text{offline}} + n \times O(N^3_{\text{red}}) + O(m^3) + O(m^2) \times N_{\text{red}}. \tag{14}$$

Numerical Results

We consider the elliptic problem in one spatial dimension

$$\begin{aligned} -(a(x,\mathbf{y})u(x,\mathbf{y})')' &= 1, \; x \in (0,1), \mathbf{y} \in \Gamma = [-1,1] \\ u(0,\mathbf{y}) &= u(1,\mathbf{y}) = 0, \end{aligned} \tag{15}$$

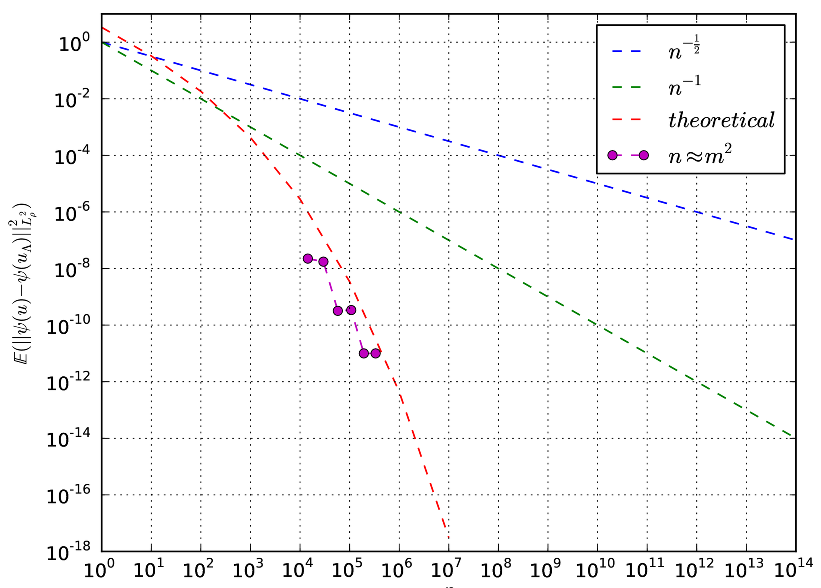
and set the diffusion coefficient to be

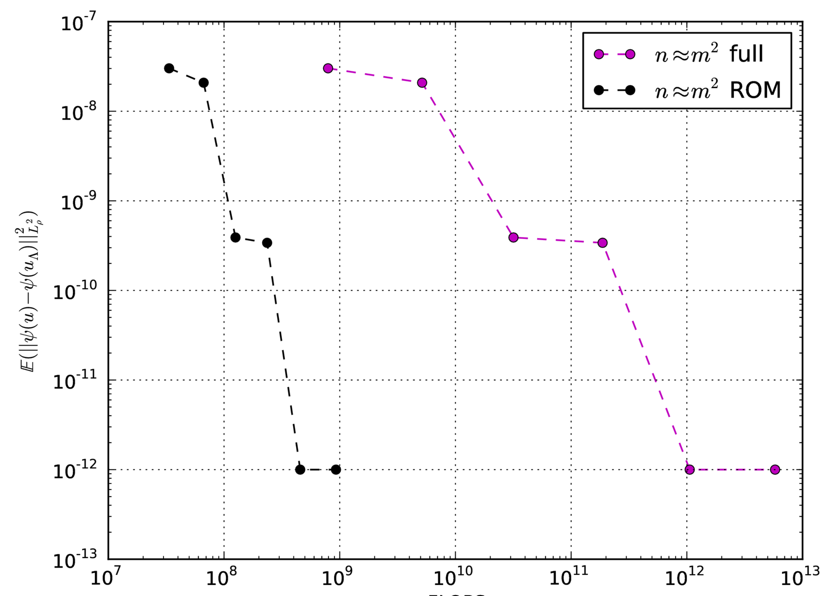
$$a(x,\mathbf{y}) = 4 + y_1 + 0.2\sin(\pi x) y_2 + 0.04 \sin(2\pi x) y_3 + 0.008 \sin(3\pi x) y_4. \tag{16}$$

In order to measure the error in our examples we will consider the quantity of interest

$$\psi(u) = \frac{1}{|D|} \int_D u dx. \tag{17}$$

- We will use a sufficiently fine discretization (h = 10000) so that the FEM error does not contribute to the total error in a meaningful fashion.





- As we can see in the above graphs we manage to maintain the accuracy of our scheme while gaining significant cost savings.