Continuous and Discontinuous Finite Element Methods for a Peridynamic Model of Mechanics

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Abstract

In contrast to classical partial differential equation models, the recently developed peridynamic nonlocal continuum model for solid mechanics is an integro-differential equation that does not involve spatial derivatives of the displacement field. As a result, the peridynamic model admits solutions having jump discontinuities so that it has been successfully applied to fracture problems. Based on a variational formulation, continuous and discontinuous Galerkin finite element methods are developed for the peridynamic model. Discontinuous discretizations are conforming for the model without the need to account for fluxes across element edges. Through a series of simple, one-dimensional computational experiments, we investigate the convergence behavior of the finite element approximations and compare the results with theoretical estimates.

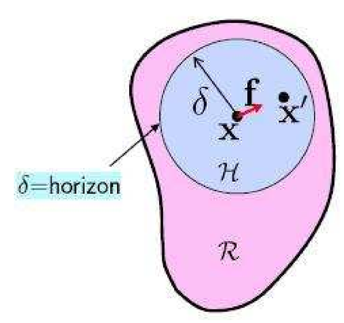
The Peridynamics Model

- **The general bond-based model**

The equation of motion at any point $\mathbf{x}$ at time t is given by:

$$\rho\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{H_\mathbf{x}} \mathbf{f}(\mathbf{u}(\mathbf{x}',t)-\mathbf{u}(\mathbf{x},t), \mathbf{x}'-\mathbf{x})\mathrm{d}V_{\mathbf{x}'} + \mathbf{b}(\mathbf{x},t) \tag{1}$$

where ρ – the mass density function,
$\mathbf{u}$ – the displacement vector field,
$H_\mathbf{x}$ – the neighborhood of $\mathbf{x}$ with radius δ,
$\mathbf{b}$ – the prescribed body force density field,
$\mathbf{f}$ – the pairwise function represents the interaction between particles.



- **A linearized peridynamics model for proportional microelastic materials** is given by the integro-differential equation

$$\rho\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{H_\mathbf{x}} c\frac{(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^3}(\mathbf{u}(\mathbf{x}',t)-\mathbf{u}(\mathbf{x},t))\mathrm{d}V_{\mathbf{x}'} + \mathbf{b}(\mathbf{x},t) \tag{2}$$

where c denotes a constant that depends not only on the material, but also on the space dimension. In one dimension, $c = \frac{18k}{5\delta^2}$, where k denotes the bulk modulus. Let $\rho = 1$ and $k = 5/18$, the steady-state, one-dimensional model setting for which (2), along with a "boundary" condition, reduces to

$$\begin{cases} \dfrac{1}{\delta^2}\displaystyle\int_{x-\delta}^{x+\delta}\frac{u(x)-u(x')}{|x-x'|}\mathrm{d}x' = b(x), & x\in\Omega \\ u(x) = g(x), & x\in\Gamma \end{cases} \tag{3}$$

where, $\Omega = (\alpha,\beta)$, $\Omega' = (\alpha-\delta,\beta+\delta)$, $\Gamma = \overline{\Omega'}\backslash\Omega = [\alpha-\delta,\alpha]\cup[\beta,\beta+\delta]$.

Well-Posedness of the Variational Problem

Let $S(\Omega')$ denote a Banach space, such that $L^2(\Omega')\subset S(\Omega')$. Define space $S_g(\Omega') = \{u(x)\in S(\Omega') \mid u(x) = g(x) \text{ a.e. on } \Gamma\}$ and the subspace $S_0(\Omega') = \{v(x)\in S(\Omega') \mid v(x) = 0 \text{ on } \Gamma\}$. Define

$$A(u,v) = \int_\alpha^\beta\int_{x-\delta}^{x+\delta} v(x)\frac{u(x)-u(x')}{|x-x'|}\mathrm{d}x'\mathrm{d}x,$$

and

$$F(v) = \delta^2\int_\alpha^\beta b(x)v(x)\mathrm{d}x.$$

and the "dual" space $S_0^* = \{b(x) : |||b(x)|||_* < \infty\}$. where,

$$|||b|||_* = \sup_{v\in S_0, v\neq 0}\frac{\delta^2\int_\alpha^\beta b(x)v(x)dx}{|||v|||}, \quad |||v|||^2 = A(v,v).$$

Then the variational problem has the following form: Given $g(x)\in L^2(\Gamma)$ and $b(x)\in S_0^*$, seek $u\in S_g$, such that

$$A(u,v) = F(v), \quad \forall\, v\in S_0. \tag{4}$$

By Lax-Milgram Theorem, the variational problem (4) has a unique solution and, moreover, those solutions satisfy:

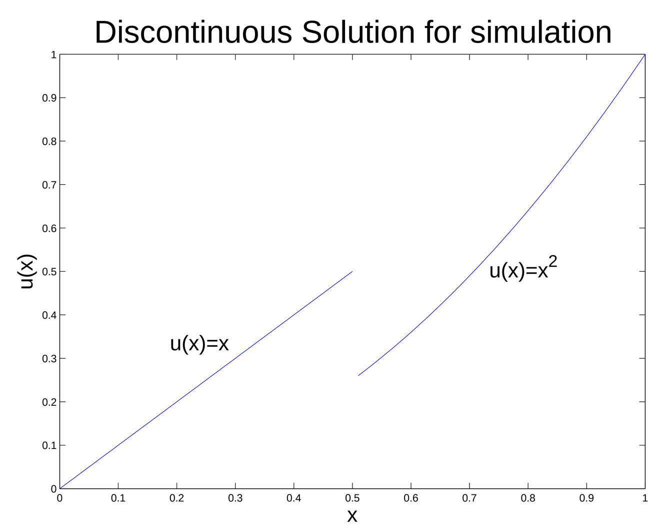
$$|||u||| \leq |||b|||_*. \tag{5}$$

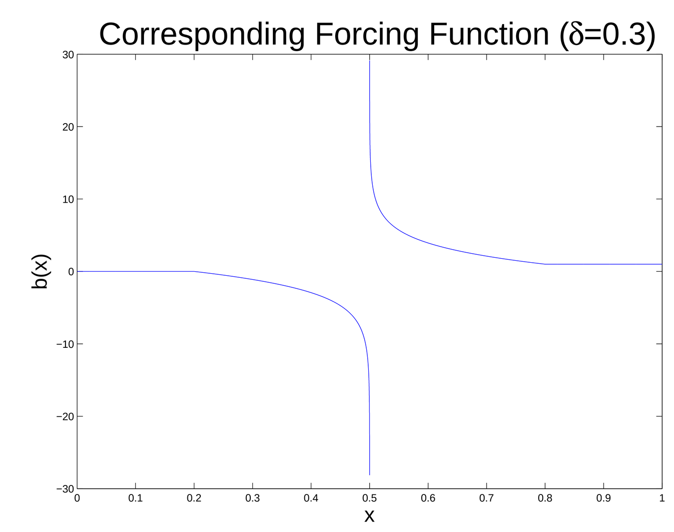
Numerical Experiments

- Smooth Solution Simulation:

	δ proportional to h			δ fixed, independent of h		
FE Space	Error(L^2)	Error(L^∞)	Error(H^1)	Error(L^2)	Error(L^∞)	Error(H^1)
CL	$O(h^2)$	$O(h^2)$	$O(h)$	$O(h^2)$	$O(h^2)$	$O(h)$
DC	$O(1)$	$O(1)$	-	$O(h)$,($\delta > h$)	$O(h)$,($\delta > h$)	-
DL	$O(h^2)$	$O(h^2)$	$O(h)$	$O(h^2)$	$O(h^2)$	$O(h)$

- Discontinuous Solution Simulation:

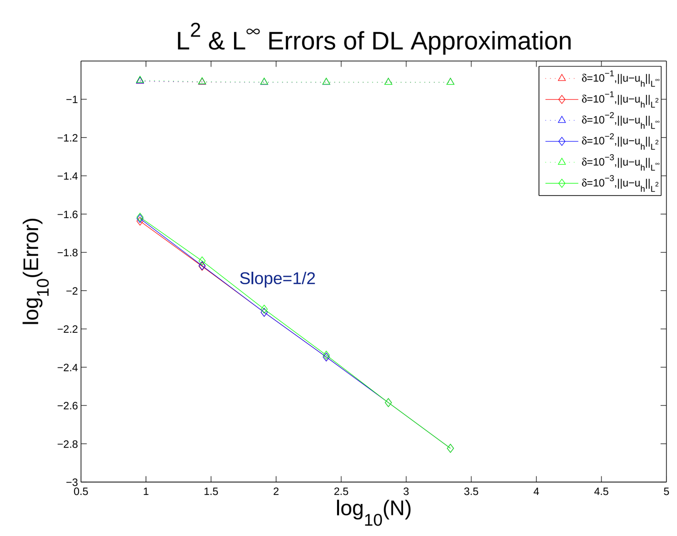


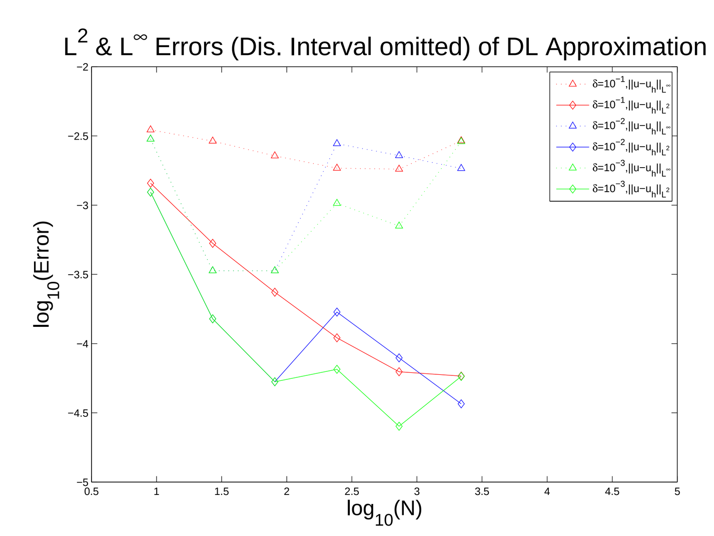


– Grid points coincides with points of discontinuity

	δ proportional to h		δ fixed, independent of h	
FE Space	Error(L^2)	Error(L^∞)	Error(L^2)	Error(L^∞)
CL	$O(h^{1/2})$	$O(h^{1/2})$	$O(h^{1/2})$	$O(h^{1/2})$
DC	$O(1)$	$O(1)$	$O(h)$,($\delta > h$)	$O(h)$,($\delta > h$)
DL	$O(h^2)$	$O(h^2)$	$O(h^2)$	$O(h^2)$

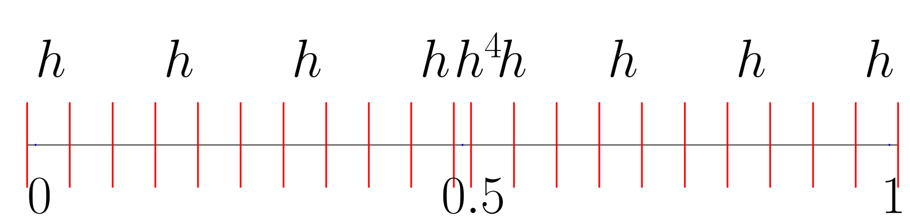
– No grid point located at discontinuous point

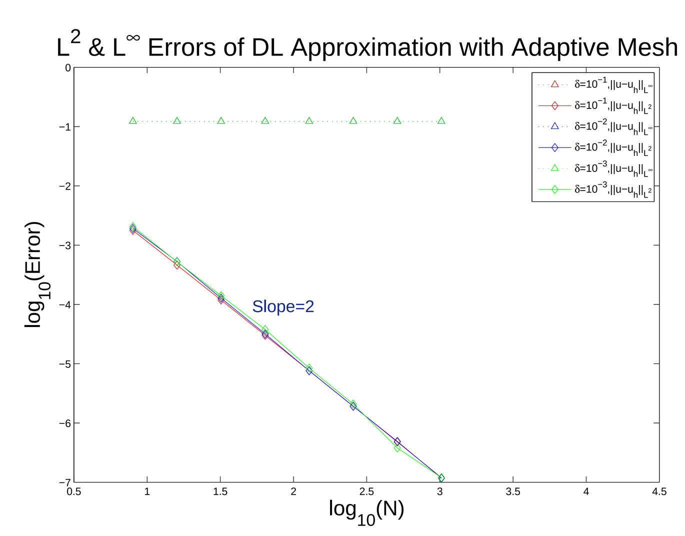


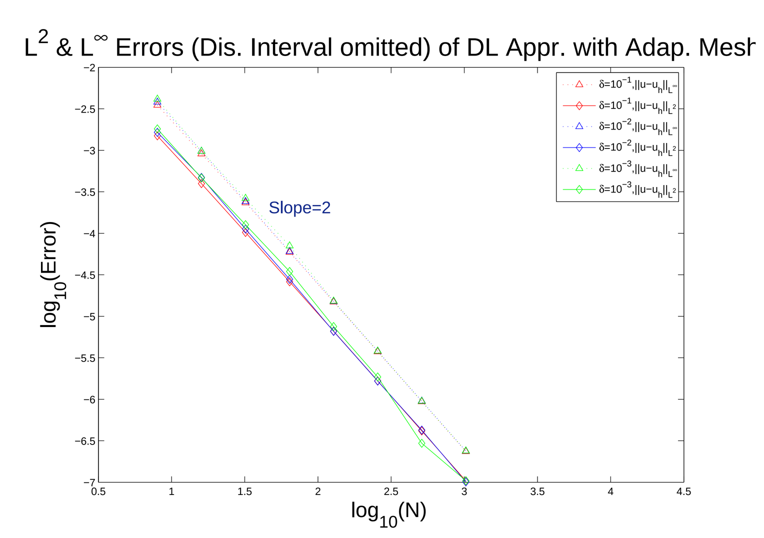


– A local grid refinement approach that recovers full accuracy

∗ Adaptive Mesh







Future Works

- Complete, rigorous analyses of errors and convergence rates and of adaptive grid refinement strategies.
- FEM Implementation in 2-3 dimensions and for nonlinear problems.
- Implementation for stochastic PD models

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